

**Probability Theory**  
**2023/24 Period IIb**  
**Instructor: Gilles Bonnet**  
**Exam**  
**21/6/2024**  
**Duration: 2 hours**

**Name:** \_\_\_\_\_

**Student number:** \_\_\_\_\_

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This exam contains 5 problems. Enter all requested information on the top of this page.

**Your answers should be written in this booklet. Avoid handing in extra paper.** In case you hand in extra paper and/or write parts of your answers in non obvious places, mark this explicitly so that it is not missed out during the grading process.

Do not use red ink.

You are **not** allowed to use any electronic devices during the exam.

You are **not** allowed to use any books, lecture notes or handwritten notes.

Each answer must be **justified**, unless explicitly stated otherwise.

Do not write on the table below.

Problem	Points	Score
1	15	
2	20	
3	14	
4	20	
5	21	
Total:	90	

Satisfying the above instructions gives you 10 free points.

1. (15 points) One deals a hand of 13 cards from a normal shuffled pack of 52 cards. What is the probability that it contains exactly one ace given that it contains exactly 2 kings?  
*Remark:* Give your answer in the form of a *fraction*, where the numerator and denominator might be product of binomial coefficients.



2. Let  $X$  be a random variable with expectation  $\mu_X$  and variance  $\sigma_X^2$ , and  $Y$  a random variable with expectation  $\mu_Y$  and variance  $\sigma_Y^2$ . Let  $\rho_{X,Y}$  be the correlation between  $X$  and  $Y$ . Express the following quantities in terms of  $\mu_X$ ,  $\mu_Y$ ,  $\sigma_X$ ,  $\sigma_Y$ , and  $\rho_{X,Y}$ .
- (a) (10 points)  $\text{Var}(X - 3Y)$ ;
  - (b) (10 points)  $\mathbb{E}((3X - 5)(2Y + 1))$ .



3. Let  $N \in \mathbb{N}$  and  $p \in (0, 1)$ .

A gambler has a starting capital of  $i\text{€}$ , with  $i \in \mathbb{N}$ . He repeatedly plays a game where he bets  $1\text{€}$  each time. With probability  $p$  he wins and he receives his money back plus an additional  $1\text{€}$ , and with probability  $1 - p$  he loses his  $1\text{€}$ .

If his capital reaches  $0\text{€}$  he is of course no longer able to play. And, if he manages to gather a capital of  $N\text{€}$ , he will go home contently.

Let  $p_i$  denote the probability that the gambler goes home contently, given that he starts with  $i\text{€}$ .

(a) (4 points) Show that for  $1 \leq i \leq N - 1$  we have the following recursive formula for  $p_i$ :

$$p_i = pp_{i+1} + (1 - p)p_{i-1}.$$

(b) (4 points) For  $1 \leq i \leq N - 1$ , express  $p_{i+1}$  in terms of  $p$  and  $p_1$ .

(c) (6 points) For  $0 \leq i \leq N$ , express  $p_i$  in terms of  $i$  and  $N$ .



4. Let  $S = \sum_{r=1}^n X_r$  be a sum of independent Bernoulli random variables  $X_r$  taking values in  $\{0, 1\}$ , where  $\mathbb{E}(X_r) = p_r$  and  $\mathbb{E}(S) = \mu > 0$ .

(a) (5 points) Show that

$$\mathbb{E}(e^{tX_r}) \leq \exp[p_r(e^t - 1)], \quad t \in \mathbb{R}, \quad r \in \{1, \dots, n\}.$$

(b) (5 points) Show that

$$\mathbb{E}(e^{tS}) \leq \exp[\mu(e^t - 1)], \quad t \in \mathbb{R}.$$

(c) (10 points) Show that

$$\mathbb{P}(S > (1 + \epsilon)\mu) \leq \exp(-\mu[(1 + \epsilon) \log(1 + \epsilon) - \epsilon]), \quad \epsilon > 0.$$

*Hint:* You might want to use Markov's inequality in combination with the bound established in the previous question, and then choose  $t$  appropriately.



5. There has been an election for the student council and there were only two candidates, let us call them  $A$  and  $B$ . The voters have cast their ballots into a sealed box. Suppose that there were  $a$  votes for  $A$  and  $b$  votes for  $B$ .

The committee in charge of counting the votes takes out the ballots one by one. The order in which they take out the ballots is completely random, because they have shaken the box really well and they don't look inside while they take the ballots out.

- (a) (6 points) Show that

$$\mathbb{P}(\text{ the last ballot they take out of the box is a vote for } A ) = \frac{a}{a+b}.$$

The committee have a blackboard at their disposal, which they've divided into two parts, the left half and the right half, separated by a vertical line. If there is a vote for  $A$  the committee make a "tally" mark on the left board and if there is a vote for  $B$  they make a "tally" mark on the right side.



(Tallies on a black board.)

We are interested in the probability that at all times during the counting process there are more tallies on the left than on the right side of the board. That is, the probability that, for all  $1 \leq i \leq a+b$ , among the first  $i$  votes there are strictly more in favour of  $A$  than in favour of  $B$ .

- (b) (15 points) Show that

$$\mathbb{P}(A \text{ is ahead during the entire counting process}) = \begin{cases} \frac{a-b}{a+b} & \text{if } a \geq b, \\ 0 & \text{otherwise.} \end{cases}$$

(Hint: Use induction on  $a$  and  $b$  with induction hypothesis "for all  $(a', b')$  with  $a' \leq a, b' \leq b$  and  $(a, b) \neq (a', b')$  the statement holds", and use part (a).)



